Two Person Games: Total Conflict with Mixed Strategies Solutions

Lesson #7

Objectives

- Recognize a game where we must use mixed strategy to find the solution.
- Find the solution to mixed strategy games.

Mixed Strategies

- A **mixed strategy** is an assignment of a probability to each pure strategy in repetitive games. It defines a probability over the strategies, and reflects that, rather than choosing a particular pure strategy, the player will randomly select a pure strategy based on the distribution given by their mixed strategy. Of course, every pure strategy is a mixed strategy which selects that particular pure strategy with probability *1* and every other one with probability *0*.

Hitter-Pitcher Duel:

- Consider the batter-pitcher is a game. The pitcher throws their pitch (fastball or curve) and the hitter has historical batting averages accordingly (his guess).

		Pitcher		
		F	C	Row min
Batter	F	.300	.200	.200
	C	.100	.500	.100
Col Max		.300	.500	No saddle

Previously

- We saw that some two-person games did not have a saddle point (equilibrium value) solution such as the game to the right where Maximin is 0 and Minimax is 2 and they are not the same.

		Columns		
		C1	C 2	RowMin
Rows	R1	2	- 3	-3
	R2	0	3	0
Column Max		2	3	

Expected Value of a Game

- The expected value of getting payoffs $a_1,a_2,\ldots,a_k$ with respective probabilities $p_1,p_2,\ldots p_k$ is
- $E[payoff] = a_1p_1+a_2p_2+\ldots+a_kp_k$
- Thus the expected value is just the weighted payoffs, where the weights are probabilities.

Expected Value Principle

- If you know that your opponent is playing a given mixed strategy, and will continue to play it regardless of what you do, you should play the strategy which has the largest Expected Value.

But what should Colin do?

- Let’s assume we need a strategy based upon probabilities that guarantees the lowest expected payoff. Let’s call the probability of Column C1 as X and the probability of Column C2 as it’s complement, 1-X.
- ROW R1: X(2)+(1-X)(-3)=-3+5X
- ROW R2: X(0)+(1-X)3=3-3X
- Set these equal and solve for X: -3+5X=3-3X or 8X=6
- X=3/4 & (1-X) = ¼
- Payoffs: Row R1: (3/4)(2)-(1/4)(-3)=3/4
- Row R2 : (0)(3/4) +(3)(1/4)=3/4
- Thus, Rose wins no more than ¾ units per game.

What if Rose plays a Mixed Strategy the same way

- Colin C1: (2)(X)+(1-X)(0)= 2x
- Colin C2: (-3)(X)+(1-X)(3) = 3- 6X
- Set these equal: 2X=3-6X or 8X=3, X=3/8
- Strategy Rose plays R1 with probability 3/8 and plays R2 with probability 5/8.
- Colin C1: (3/8)(2)+(5/8)(0)=3/4
- Colin C2: 3/8(-3)+5/8(3)=3/4

Solution

- Since the mixed strategy yielded expected values of ¾ in all cases the value of the game is ¾ when Colin plays his optimal strategy (3/4 C1)(1/4 C2) and Rose plays her optimal strategy (3/8 R1) (5/8 R2).
- The value (a saddle point) and the optimal strategies are called the solution of the game.
- Note: if the game has a saddle point with “pure strategies” then this method WILL NOT produce optimal strategies. **Thus, we check “pure strategy” solution techniques first.**

Shorthand Method: Method of Oddments

- Colin
  Row Rose Rose
- C1 C2

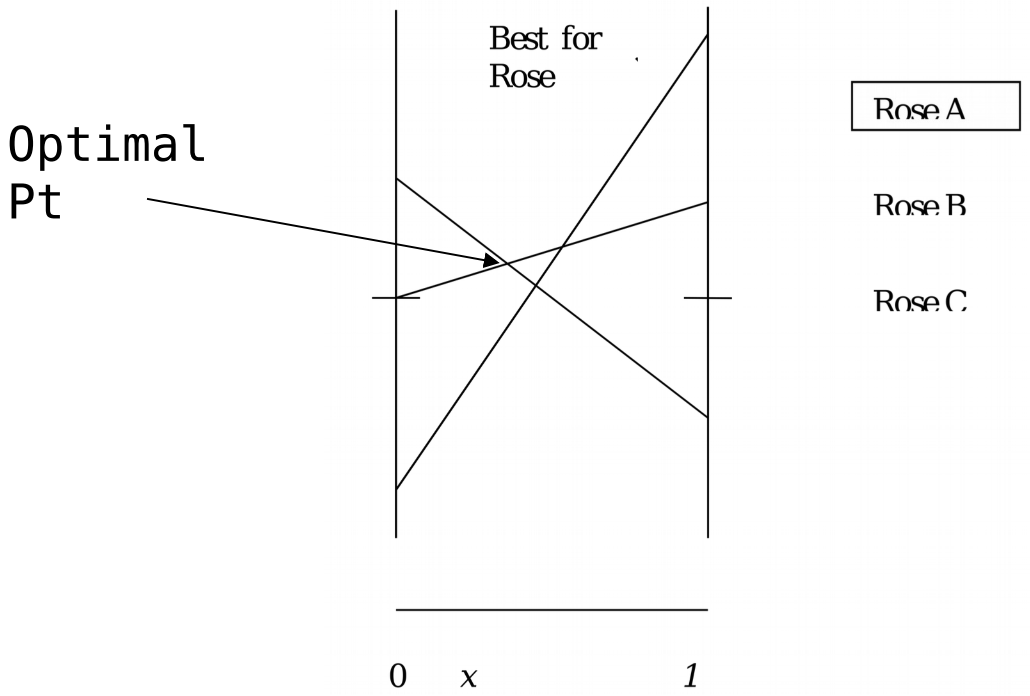
differences oddments probabilities

- ROSE R1 2 -3
  2-(-3)=5 3 3/8
- R2 0 3
  0-3=-3 5 5/8
- Column Differences 2-0=2 -3-3=-6
- Column Oddments 6 2
- Column Probabilities 6/8 2/8

Let’s look at a larger game

		Colin		
		A	B	Row Min
Rose	A	2	-3	-3
	B	0	2	0
	C	-5	10	-5
Column Max		2	10	

- This game does not have a saddle point. Dealing with more than 2 alternatives affects the differences, so we first check for dominance to reduce the size of the game or we have use another method to assist us. Note: there is not dominance here.
- This method starts with a graph.



Best for
Rose
Optimal
Pt
Rose A
Rose B
Rose C
0
x
1

- We see from the graph that the optimal point is the intersection of Rose A and Rose B, so we can definitely never choose Rose C. We eliminate it.

		Colin	
		A	B
Rose	A	2	-3
	B	0	2

Solve the Subgame

		Colin		
		A	B	
Rose	A	2	-3	2/7
	B	0	2	5/7
		5/7	2/7	

Rose plays A with probability 2/7 and B with Probability 5/7 and should never play C.
The value of the game is 4/7.

Consider the following game:

	#1	#2	#3	Rowmin
#1	1	1	10	1
#2	2	3	-4	-4
ColMax	2	3	10	No Saddle

- Column #1 dominates Column #2 since every entry in Col #1 $\leq$ every corresponding entry in Col 2.
- The game is reduced to:

	#1	#3	Rowmin
#1	1	10	1
#2	2	-4	-4
ColMax	2	10	No saddle

We can now more easily solve this new game

- p+(1-p)*10=2p+(1-p)*(-4)
- p -10p +10=2p+4p-4
- -9 p +10 = 6 p -4
- 14 = 15 p
- p = 14/15 and (1-p) = 1/15
- Value of Game: <u>24/15</u>

Games bigger than *2 x n* or *n x 2*.

- Consider a game where each player has 3 alternative strategies, as below:

		Colin		
		A	B	C
Rose	A	1	2	2
	B	2	1	2
	C	2	2	0

Methodology

- Check for a saddle point solution (there is none).
- Check for Dominance (there is no dominance)
- We will not use a graph since each player has more than 2 alternatives.
- We use a Method called “Equalizing Expectations”.
- This method fails is the solution involves a 1 x 1 or 2 x 2 subgame. So if this method fails look for a 2 x 2 solution by graphing all possible 2 x 3 subgames.

Equalizing Expectations

- Assume Colin plays A,B, and C with probabilities $(x,y,1-x-y)$
- Rose A: $x(1)+y(2)+(1-x-y)(2)=2-x$
- Rose B: $x(2)+y(1)+(1-x-y)(2)=2-y$
- Rose C: $x(2)+y(2)+(1-x-y)(0)=2x+2y$
- Set them all equal: $2-x=2-y=2x+2y$.

- This simplifies to
- x-y=0
- 2x+3y=2
- We solve by substitution and get x=y=2/5
- The three probabilities are (x,y,1-x-y) = (2/5,2/5,1/5)
- Value of the game 2/5(1)+2/5(2)+1/5(1)=8/5